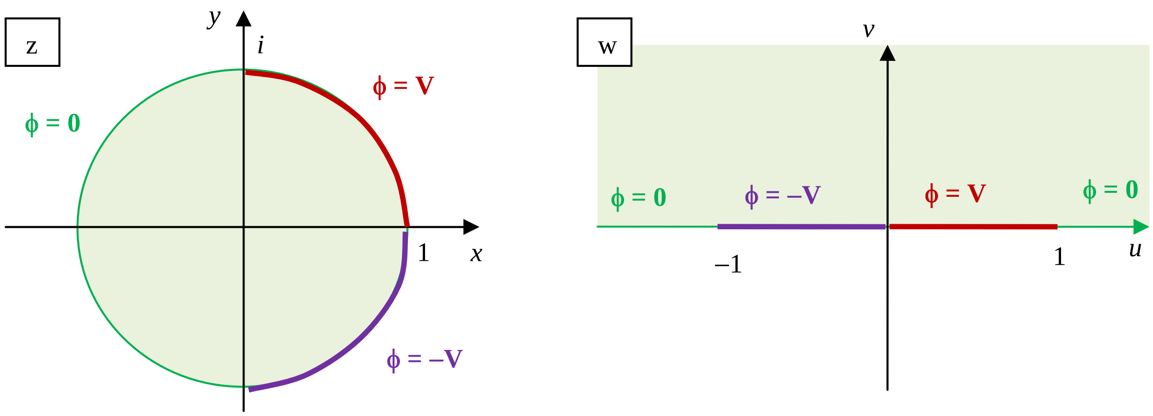
Math 756 • Final Exam • Victor Matveev • December 2013

1) (25pts)



a) Pick the following points to map: $w(1) = 0,\ w(i) = 1,\ w(-1) = \infty,\ z = z_4,\ w = w_4$:

$$\frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(1-z)(-1-i)}{(1-i)(-1-z)} = \frac{(1-z)(1+i)}{(1+z)(1-i)} = i\,\frac{1-z}{1+z}$$

$$\frac{(w_1 - w_4)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w_4)} = \frac{(0-w)(\infty-1)}{(0-1)(\infty-w)} = w$$

$$\Rightarrow \boxed{w = i\,\frac{1-z}{1+z}}$$

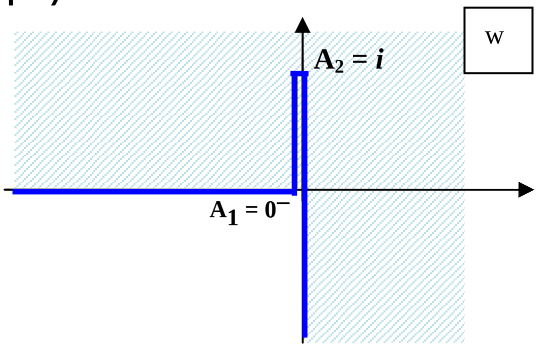
Let's invert: $w(1+z) = i\,(1-z) \ \Rightarrow\ z\left(w+i\right) = i - w \ \Rightarrow \boxed{z = \frac{i-w}{i+w}}$

b) Solution satisfying the boundary conditions is:

$$\phi(w) = \frac{V}{\pi}\arg\left(w-1\right) - \frac{2V}{\pi}\arg w + \frac{V}{\pi}\arg(w-1)$$

c) Thus, the solution is $\Phi(z) = \phi\left(z(w)\right) = \boxed{\frac{V}{\pi}\arg\left(i\,\frac{1-z}{1+z} - 1\right) - \frac{2V}{\pi}\arg\left(i\,\frac{1-z}{1+z}\right) + \frac{V}{\pi}\arg\left(i\,\frac{1-z}{1+z} - 1\right)}$

2) (20pts)



From the picture it is clear that $\alpha_1 = \frac{1}{2},\ \alpha_2 = 2$, therefore

$$\frac{dw}{dz} = \gamma z^{\frac{1}{2}-1}(z-1)^{2-1} = \gamma \frac{z-1}{z^{1/2}} = \gamma\left(z^{1/2} - z^{-1/2}\right)$$

$$\Rightarrow\ w = \gamma\left(\frac{2}{3}z^{3/2} - 2z^{1/2}\right) + A = \frac{2\gamma}{3}\left(z^{3/2} - 3z^{1/2}\right) + A = \tilde{\gamma} z^{1/2}(z-3) + A$$

$$\left.\begin{array}{l} w(0) = 0 \ \Rightarrow\ A = 0 \\ w(1) = i \ \Rightarrow\ -2\tilde{\gamma} = i \ \Rightarrow\ \tilde{\gamma} = -\dfrac{i}{2} \end{array}\right\} \Rightarrow \boxed{w(z) = -\frac{i}{2} z^{1/2}(z-3)}$$

---

3) (30pts) Consider a **clockwise** vortex of strength Γ centered at x_o=2, near a cylindrical obstacle of radius 1 centered at the origin (as usual, assume ideal fluid flow):

a) Apply the Milne-Thompson Theorem, noting that $a = 1,\ z_o = 2:\ g_o(z) = \frac{i\Gamma}{2\pi}\log(z-2)$:

$$g(z) = g_o(z) + \overline{g_o(1/\bar{z})} = \frac{i\Gamma}{2\pi}\log(z-2) + \overline{\frac{i\Gamma}{2\pi}\log\left(\frac{1}{\bar{z}} - 2\right)} = \frac{i\Gamma}{2\pi}\log(z-2) - \frac{i\Gamma}{2\pi}\log\left(\frac{1-2z}{z}\right)$$

$$= \frac{i\Gamma}{2\pi}\left[\log(z-2) - \underbrace{\log(1-2z)}_{\log(z-1/2)+const} + \log z\right] = \frac{i\Gamma}{2\pi}\left[\log(z-2) - \log\left(z - \frac{1}{2}\right) + \log z\right] + const$$

Thus, the image with respect to the cylinder represents two vortices of opposite spin, both centered inside the cylinder, at z=0 and at z=1/2. This is always the case since the total vorticity should not be changed by the obstacle, and the method of images only adds singularities outside of the physical domain, within the obstacle

---

b) Complex velocity:

$$\overline{U} = \frac{dg}{dz} = \frac{i\Gamma}{2\pi}\left[\frac{1}{z-2} - \frac{1}{z-1/2} + \frac{1}{z}\right] = \frac{i\Gamma}{2\pi}\frac{z(z-1/2)-z(z-2)+(z-2)(z-1/2)}{z(z-1/2)(z-2)} = \frac{i\Gamma}{2\pi}\frac{z^2-z+1}{z(z-1/2)(z-2)}$$

Stagnation points: $U = 0 \;\Rightarrow\; z^2 - z + 1 = 0 \;\Rightarrow\; z = \dfrac{1+(1-4)^{1/2}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$

These points are on the surface of the obstacle, at $\theta = \pm\dfrac{\pi}{3}$

---

c) Blasius Theorem: $\overline{F} = \dfrac{i\rho}{2}\oint_{\partial B}\left(\dfrac{dg}{dz}\right)^2 dz = -\dfrac{i\rho}{2}\left(\dfrac{\Gamma}{2\pi}\right)^2 \oint_{|z|=1}\left[\dfrac{1}{z-2} - \dfrac{1}{z-1/2} + \dfrac{1}{z}\right]^2 dz$

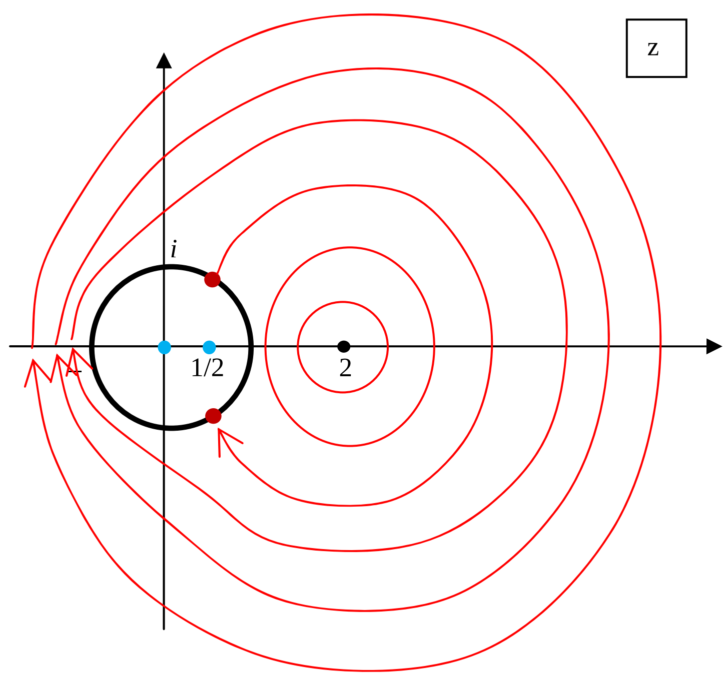
Expand the square in the integrand but only keep cross-terms that have simple poles at $z = 0$ and at $z = 1/2$ (since z=2 is outside the integration contour), and find the residues:

$$\overline{F} = -\frac{i\rho}{2}\left(\frac{\Gamma}{2\pi}\right)^2 \oint_{|z|=1}\left[\frac{2}{\mathbf{z}(z-2)} - \frac{2}{\mathbf{(z-1/2)}\,(z-2)} - \frac{2}{\mathbf{z\,(z-1/2)}}\right]dz$$

$$= \frac{\rho\Gamma^2}{2\pi}\left[\frac{1}{0-2} - \frac{1}{1/2-2} - \frac{1}{0-1/2} - \frac{1}{1/2}\right] = \frac{\rho\Gamma^2}{2\pi}\left[-\frac{1}{2} + \frac{2}{3} + 2 - 2\right] = \frac{\rho\Gamma^2}{12\pi} \;\Rightarrow\; \boxed{F = \frac{\rho\Gamma^2}{12\pi}}$$

Force > 0 (rightward) since velocity is the strongest (and hence pressure the lowest) closer to the vortex

.v



4) (30pts)

i) Map vertices of the square: $w(\pm 1) = \pm 1 + \frac{1}{\pm 1} = \pm 2,\;\; w(\pm i) = \pm i + \frac{1}{\pm i} = \pm i \mp i = 0$

Map centers of the sides: $w\left(\frac{\pm 1 \pm i}{2}\right) = \frac{\pm 1 \pm i}{2} + \frac{2}{\pm 1 \pm i} = \frac{\pm 1 \pm i}{2} \pm 1 \mp i = \pm \frac{3}{2} \mp \frac{i}{2}$

Map is conformal at z= $\pm i$ $\Rightarrow$ Curves intersect at right angles at $w(\pm i) = 0$

---

ii) Map is not conformal at $z_o = \pm 1$ because $w'(\pm 1) = 0$; however $w''(\pm 1) \neq 0$ (see below), therefore we can expand to second order around $z_o = \pm 1$:

$$w''(z) = \frac{d^2}{dz^2}\left(z + \frac{1}{z}\right) = \frac{d}{dz}\underbrace{\left(1 - \frac{1}{z^2}\right)}_{w'(\pm 1)=0} = \frac{2}{z^3} \quad \Rightarrow w''(\pm 1) = \pm 2 \neq 0$$

$$\Rightarrow w(z) \approx w(z_o) + \underbrace{w'(z_o)}_{=0}(z \mp 1) + \frac{w''(z_o)}{2}(z - z_o)^2 \quad \Rightarrow \boxed{W = \gamma Z^2} \text{ where } W \equiv w - w_o,\; Z \equiv z - z_o,\; \gamma = \pm 1$$

$\Rightarrow$ Angles are **doubled** at images of ± 1, and right angles unwrap into smooth curves ($\Delta\theta = \pi$)

$\Rightarrow$ The images of the sides of the square are tangent to each other at $w(\pm 1) = \pm 2$ (see Figure below)

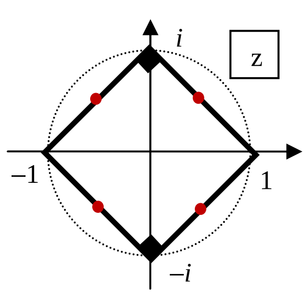
---

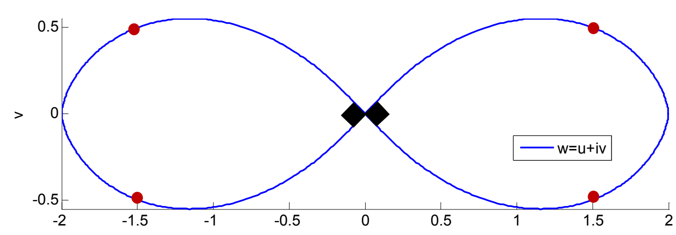
iii) Since the circle maps to a real interval $[-2,2]$, by the open set mapping theorem the interior (as well as the exetrior) of the circle map to $\mathbb{C}/[-2,2]$ (open set = complex plane excluding [-2,2])

Next, note that the center of the square maps to infinity (w(0)=∞)

$\Rightarrow$ the **interior** of the square maps to the region **exterior** to the figure-8 region in the plot

$\Rightarrow$ the region enclosed between the square and the circle maps to the **interior** of the figure-8





5) (15pts extra credit) This map is easier, since this square is tangent to the circle at the points of interest z=±1 and z=±*i*, therefore at these points the images of the sides of the square are tangent to the image of the circle, which is a horizontal real interval (note the horizontal tangent at w=±2 and w=0). At the images of the four vertices, w(±1±i)=±3/2 ± i/2 (note they map to the same points as the mid-points labeled in red in problem 4), the map is conformal, so the angle is preserved and equals π/2